THE PROPAGATION OF COMPRESSIVE-SHEAR PERTURBATIONS IN A NONLINEARLY ELASTIC MEDIUM

(RASPROSTRANENIE SZHIMAIUSHCHE-SDVIGAIUSHCHIKH VOZMUSHCHENII V NELINEINO-UPRUGOI SREDE)

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V.S.ANTSIFEROV and Kh.A.RAKHMATULIN
(Moscow)

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1. Consider a homogeneous medium at rest occupying the lower half-space in Fig.1. We introduce a rectangular system of Lagrangean coordinates $Oxyz$ and consider the stress and small-strain tensors

$$\left\| \begin{matrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ X_z & Y_z & Z_z \end{matrix} \right\|, \qquad \left\| \begin{matrix} \varepsilon_{xx} & {}^1/_2\,\varepsilon_{xy} & {}^1/_2\,\varepsilon_{xz} \\ {}^1/_2\,\varepsilon_{xy} & \varepsilon_{yy} & {}^1/_2\,\varepsilon_{yz} \\ {}^1/_2\,\varepsilon_{xz} & {}^1/_2\,\varepsilon_{yz} & \varepsilon_{zz} \end{matrix} \right\|$$

with first invariants

$$P = {}^1/_3\,(X_x + Y_y + Z_z), \qquad \varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

and second invariants

$$T = {}^1/_2\sqrt{2}\sqrt{(X_x - Y_y)^2 + (Y_y - Z_z)^2 + (Z_z - X_x)^2 + 6\,(X_y^2 + X_z^2 + Y_z^2)}$$

$$\Gamma = {}^1/_2\sqrt{2}\sqrt{(\varepsilon_{xx} - \varepsilon_{yy})^2 + (\varepsilon_{yy} - \varepsilon_{zz})^2 + (\varepsilon_{zz} - \varepsilon_{xx})^2 + {}^3/_2(\varepsilon_{xy}^2 + \varepsilon_{xz} + \varepsilon_{yz}^2)}$$

respectively.

Suppose that the medium in question can be described by a model of a nonlinearly elastic body with the properties

$$P = \varphi(\varepsilon), \qquad D_\sigma = 6QD_\varepsilon\,, \qquad T = f(\Gamma), \qquad Q(\Gamma) \equiv \frac{f(\Gamma)}{6\Gamma}$$

In these formulas D_σ and D_ε are the stress and strain deviators $\varphi(\varepsilon)$ and $f(\Gamma)$ are known functions.

Suppose that at the initial moment of time $t = 0$ all particles lying on the surface of the medium $(x = 0)$ are given one and the same velocity $\mathbf{V}_0(V_{x0}, V_{y0}, 0)$, which then remains constant (Fig.1), where $V_{y0} \geqslant 0$. Plane uniform motion takes place within the medium, so that all parameters of motion depend only on x and t. We introduce a particle-velocity vector $\mathbf{V}(V_x, V_y, 0)$ and a particle-displacement vector $U(u, v, 0)$. By virtue of symmetry of the problem, we have

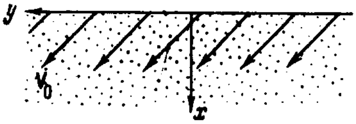

Fig. 1

$$\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{xy} = \frac{\partial v}{\partial x} \tag{1.1}$$

$$\varepsilon = \varepsilon_{xx}, \qquad \Gamma = \sqrt{\varepsilon^2 + {}^3/_4\, \varepsilon_{xy}{}^2} \tag{1.2}$$

Hence

$$X_x - \varphi(\varepsilon) = 4Q(\Gamma)\,\varepsilon, \qquad X_y = 3Q(\Gamma)\,\varepsilon_{xy} \tag{1.3}$$

Since $V_x = \partial u / \partial t$ and $V_y = \partial v / \partial t$, we obtain from (1.1) that

$$\frac{\partial V_x}{\partial x} = \frac{\partial \varepsilon}{\partial t}, \qquad \frac{\partial V_y}{\partial x} = \frac{\partial \varepsilon_{xy}}{\partial t} \tag{1.4}$$

The equations of motion are of the form [1]

$$\rho_0 \frac{\partial V_x}{\partial t} = \frac{\partial X_x}{\partial x}, \qquad \rho_0 \frac{\partial V_y}{\partial t} = \frac{\partial X_y}{\partial x} \tag{1.5}$$

where ρ_0 is the density of the medium at rest. Equations (1.2) to (1.5) form a closed system which must be solved for the conditions

$$V_x = V_{x0}, \qquad V_y = V_{y0} \qquad \text{for } x = 0 \tag{1.6}$$

$$V_x = 0, \quad V_y = 0, \quad \varepsilon = 0, \quad \varepsilon_{xy} = 0 \qquad \text{for } x = \infty \tag{1.7}$$

Making use of (1.3) and eliminating X_x and Y_y from (1.5), we obtain

$$\frac{\partial V_x}{\partial t} - a\frac{\partial \varepsilon}{\partial x} - b\frac{\partial \varepsilon_{xy}}{\partial x} = 0, \qquad \frac{\partial V_y}{\partial t} - b\frac{\partial \varepsilon}{\partial x} - c\frac{\partial \varepsilon_{xy}}{\partial x} = 0 \tag{1.8}$$

Here

$$a = \frac{1}{\rho_0}\left(\frac{d\varphi}{dE} + 4Q(\Gamma) + \frac{4\varepsilon^2}{\Gamma}\frac{dQ}{d\Gamma}\right), \qquad b = \frac{1}{\rho_0}\frac{3\varepsilon\varepsilon_{xy}}{\Gamma}\frac{dQ}{d\Gamma}$$

$$c = \frac{1}{\rho_0}\left(3Q(\Gamma) + \frac{9}{4}\frac{\varepsilon_{xy}{}^2}{\Gamma}\frac{dQ}{d\Gamma}\right) \tag{1.9}$$

It can easily be seen that the characteristics of the system (1.4), (1.8) may be found from Equation

$$(dx / dt)^2 = {}^1/_2\,[a + c \pm \sqrt{(a - c)^2 + 4b^2}]$$

For the existence of four families of characteristics we have the condition $a + c > \sqrt{(a - c)^2 + 4b^2}$, or

$$\left(Q + \frac{3\varepsilon_{xy}{}^2 Q'}{4\Gamma}\right)\varphi' + 4Q(Q + Q') > 0 \tag{1.10}$$

Since $\varphi(\varepsilon)$ and $Q(\Gamma)$ are in essence independent functions, it follows that the inequality $Q + \Gamma Q' > 0$ is necessary if (1.10) is to hold. It can be shown that it would also be a sufficient condition for (1.10), if $\varphi' > 0$ (which is natural). We make the assumption that the inequality $Q + \Gamma Q' > 0$ always holds. It is easily seen that a necessary and sufficient condition for this inequality to hold is that the function $T(\Gamma)$ increases monotonically.

With these assumptions the system has four families of characteristics

$$\left(\frac{dx}{dt}\right)_1^2 = \pm\frac{a + c + \sqrt{(a - c)^2 + 4b^2}}{2}, \qquad \left(\frac{dx}{dt}\right)_2^2 = \pm\frac{a + c - \sqrt{(a - c)^2 + 4b^2}}{2} \tag{1.11}$$

It will be noted that in the case of pure compression (or expansion) without shear ($V_y = 0$, $\varepsilon_{xy} = 0$) the system (1.4), (1.8) has two families of characteristics given by the formula for $(dx/dt)_1$. In case of pure shear

($V_x \equiv 0,\ \varepsilon \equiv 0$) this system has two families of characteristics given by the formula for $(dx/dt)_2$.

For this reason we shall call the characteristics $(dx/dt)_1$ and $(dx/dt)_2$ compression and shear characteristics, respectively. It can easily be shown that

$$\left|\left(\frac{dx}{dt}\right)_1\right|^2 > \left|\left(\frac{dx}{dt}\right)_2\right|^2$$

2. In view of the self-similarity of the problem under discussion we conclude that the parameters V_x, V_y, ε, ε_{xy}, X_x and X_y depend only on the variable $\xi = x/t$. In this case the system (1.4), (1.8) is easily reduced to the form

$$\frac{dV_x}{d\xi} + \xi\frac{d\varepsilon}{d\xi} = 0, \qquad \frac{dV_y}{d\xi} + \xi\frac{d\varepsilon_{xy}}{d\xi} = 0 \tag{2.1}$$

$$(\xi^2 - a)\frac{d\varepsilon}{d\xi} = b\frac{d\varepsilon_{xy}}{d\xi}, \qquad (\xi^2 - c)\frac{d\varepsilon_{xy}}{d\xi} = b\frac{d\varepsilon}{d\xi}$$

Conditions (1.6) and (1.7) become

$$V_x = V_{x0}, \qquad V_y = V_{y0} \qquad \text{for } \xi = 0 \tag{2.2}$$

$$V_x = 0, \qquad V_y = 0, \qquad \varepsilon = 0, \qquad \varepsilon_{xy} = 0 \qquad \text{for } \xi = \infty$$

The motion in a pure-compression (or expansion) wave adjacent to the zone at rest is described by equations obtained from (2.1) with $V_y \equiv 0$, $\varepsilon_{xy} \equiv 0$

$$\frac{dV_x}{d\xi} + \xi\frac{d\varepsilon}{d\xi} = 0, \quad (\xi^2 - a_0)\frac{d\varepsilon}{d\xi} = 0 \qquad \left(a_0 = a_0(\varepsilon) = \frac{1}{\rho_0}[\varphi' + 4Q(|\varepsilon|)\,Q'(|\varepsilon|)]\right) \tag{2.3}$$

The general solution to the system (2.3) defines a constant flow

$$V_x = V_{x1} = \text{const}, \qquad \varepsilon = \varepsilon_1 = \text{const}$$

the zone of which is separated from the zone at rest by a compression (or expansion) shock-wave. Taking into account (1.3), we can write the conditions on the shock-wave [1] in the form

$$V_{x1} + D\varepsilon_1 = 0, \qquad \rho_0 D V_{x1} = -(X_{x1} - X_{x0})$$

$$X_{x0} = \varphi(0), \qquad X_{x1} = \varphi(\varepsilon_1) + 4Q(|\varepsilon_1|)\,\varepsilon_1 \tag{2.4}$$

Here D is the (constant) velocity of the shock-wave, the parameters in the zone of constant flow behind the shock-wave being denoted by the suffix 1 The particular solution to the system (2.3) (a centered wave) is

$$\xi = \sqrt{a_0(\varepsilon)}, \qquad V_x = -\int_0^{\varepsilon} \sqrt{a_0(\varepsilon)}\, d\varepsilon$$

If for compressive (or tensile) deformations $da_0/d|\varepsilon| < 0$, then a centered compression (or expansion) wave will propagate through the undisturbed medium. If however $da_0/d|\varepsilon| > 0$, a compression (or expansion) shock-wave will travel in front. If $a_0 \equiv \text{const}$, a compression (or expansion) shock-wave travels in front also. For simplicity we shall not consider cases when $da_0/d|\varepsilon|$ changes sign.

Consider now a chear-compression (or expansion) wave propagating through a region of pure compression (or expansion). The general solution to the system (2.1) defines the constant flow

$$V_x = V_{x2} = \text{const}, \qquad V_y = V_{y2} = \text{const}, \quad \varepsilon = \varepsilon_2 = \text{const}, \qquad \varepsilon_{xy} = \varepsilon_{xy2} = \text{const}$$

The region of constant flow accompanied by shear can be separated by a shock-wave from the region $\varepsilon_{xy} = 0$, $V_y = 0$. We can write down the conditions on this shock-wave, denoting the parameters in front of and behind the wave, respectively, by the suffixes 1 and 2 $(V_{x2} = V_{x0},\ V_{y2} = V_{y0})$. Suppose that the wave velocity relative to fixed space is b and the equation of the wave is

$$x = x^* = Ct$$

Then from the relations

$$\int_0^t b\,dt = x^* + u_1(x^*, t), \quad \varepsilon = \frac{\rho_0}{\rho} - 1, \quad u_2(x^*, t) = u_1(x^*, t), \quad v_2(x^*, t) = 0$$

we can easily obtain

$$(b - V_{x1})\rho_1 = \rho_0 C, \qquad V_{y0} = -C\varepsilon_{xy2}, \qquad V_{x0} + C\varepsilon_2 = V_{x1} + C\varepsilon_1 \tag{2.5}$$

and the relations [1]

$$\rho_1 (b - V_{x1}) V_{y0} = -X_{y2}, \quad \rho_1 (b - V_{x1})(V_{x0} - V_{x1}) = -(X_{x1} - X_{x2})$$

assume the form

$$\rho_0 C V_{y0} = -X_{y2}, \quad \rho_0 C (V_{x0} - V_{x1}) = -(X_{x1} - X_{x2}) \tag{2.6}$$

Conditions (2.5) and (2.6) must be supplemented by Equations (1.3)

$$X_{x2} = C(\varepsilon_2) + 4\varepsilon_2 Q(\Gamma_2), \qquad X_{y2} = 3\varepsilon_{xy2} Q(\Gamma_2), \quad \Gamma_2 = \sqrt{\varepsilon_2^2 + {}^3/_4\, \varepsilon^2_{xy2}} \tag{2.7}$$

For the shear-compression (expansion) shock-wave to be stable it is essential [2] for its velocity to be not less than the velocity of small shear perturbations ahead of the wave and not greater than the velocity of small shear and compression perturbations behind the wave. Simple computations show that the shock-wave will be stable if the inequality

$$Q'(\Gamma_2) \geqslant 0 \tag{2.8}$$

holds.

Consider now the particular solution of the system (2.1). Multiplying together the last two of Equations (2.1) and cancelling the product $(d\varepsilon/d\xi)(d\varepsilon_{xy}/d\xi)$, assumed to be no-zero, we obtain $(\xi^2 - a)(\xi^2 - c) = b^2$.

From this we have that either

$$\xi = [{}^1/_2 (a + c + \sqrt{(a - c)^2 + 4b^2})]^{1/2} \tag{2.9}$$

or

$$\xi = [{}^1/_2 (a + c - \sqrt{(a - c)^2 + 4b^2})]^{1/2} \tag{2.10}$$

It can easily be shown that (2.9) cannot provide a solution to the problem. Indeed, a comparison of (2.9) and (1.11) shows that for the advancing compression characteristics $dx/dt = \xi = x/t$, i.e. in the zone of solution (2.9) the characteristics (2.9) are (in the xt-plane) rays emanating from the origin of coordinates. Consequently, the ray separating the regions of pure compression and shear-compression in the xt-plane is a compression characteristic. But this means that the front of the shear-compression shock-wave leaves behind the waves of small shear perturbations originating from the surface of the medium, since

$$\left|\left(\frac{dx}{dt}\right)_1\right| > \left|\left(\frac{\partial x}{\partial t}\right)_2\right|$$

This is not possible.

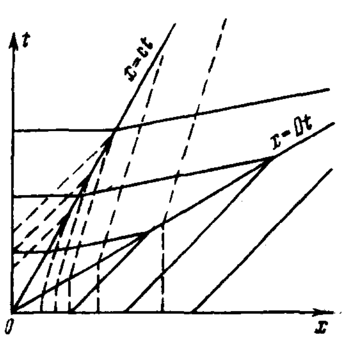

Fig. 2

Consequently the particular solution is given by Formula (2.10) together with the first three of equations (2.1). The same particular solution (2.9) corresponds to a leading pure compression wave. By analogous reasoning we conclude that in the zone of the particular solution the shear characteristics are rectilinear rays in the xt-plane. For this reason this solution is called a centered shear-wave. Ahead of the centered shear-wave there must be a region of constant flow, since the proximity of centered compression and shear waves requires the shear and compression characteristics to coincide on their boundary, which is not possible.

3. It follows from the above that the form of the solution depends on the form of the functions $a_0(\varepsilon)$ and $Q(\Gamma)$. If we do not consider the cases when $a_0 = \mathrm{const}$, $Q \equiv \mathrm{const}$ in view of their simplicity, then in all there can be four variants of the ensuing motion. We shall now consider these, assuming, in order to be specific, that $V_{x0} > 0$ (compression):

Variant 1

$$da_0 / d\,|\varepsilon| > 0, \qquad dQ / d\Gamma > 0$$

A pure compression shock-wave propagates throughout the undisturbed medium; behind this wave there is a zone of constant flow, through which a compression-shear shock-wave travels [7 and 8]. Between the second shock wave and the surface of the medium there is a zone of constant flow. The motion pattern in the xt-plane is shown in Fig.2.

The solution is of the form

$$\begin{aligned}
&\varepsilon = 0, \quad \varepsilon_{xy} = 0, \quad V_x = 0, \quad V_y = 0, \quad \text{for } x > Dt \\
&\varepsilon = \varepsilon_1, \quad \varepsilon_{xy} = 0, \quad V_x = V_{x1}, \quad V_y = 0, \quad \text{for } Dt > x > Ct \\
&\varepsilon = \varepsilon_2, \quad \varepsilon_{xy} = \varepsilon_{xy2}, V_x = V_{x0}, \quad V_y = V_{y0} \quad \text{for } Ct > x \geqslant 0
\end{aligned} \tag{3.1}$$

Here V_{x0} and V_{y0} are known quantities. In order to determine the constants ε_1, ε_2, ε_{xy2}, V_{x1}, D, and C we have the system of equations (2.4) to (2.7). The above solution may be used for the experimental determination of the functions $\varphi(\varepsilon)$ and $Q(\Gamma)$. From (2.4) to (2.7) we easily obtain

$$\varphi(\varepsilon_2) = X_{x2} - \frac{4}{3\rho_0}\frac{X_{y2}^2}{V_{y0}^2}E_2, \qquad Q(\Gamma_2) = \frac{1}{3\rho_0}\frac{X_{y2}^2}{V_{y0}^2}, \qquad \Gamma_2 = \sqrt{\varepsilon_2^2 + {}^3/_4\,\varepsilon_{xy2}^2}$$

$$\varepsilon_{xy2} = \frac{\rho_0 V_{y0}^2}{X_{y2}}, \qquad \varepsilon_2 = \frac{\rho_0 V_{y0}^2 (X_{x2} - X_{x0} + \rho_0 D V_{x0}) - X_{y2}^2 V_{x0}^2 + V_{y0} V_{y2}(X_{x2} - X_{x0})}{X_{y2}(\rho_0 D V_{y2} + X_{y2})} \tag{3.2}$$

We see from Formulas (3.2) that by measuring the velocity D of the leading shock-wave and the fcur quantities $(V_{x0}, V_{y0}, X_{x2}, X_{y2})$, on the surface

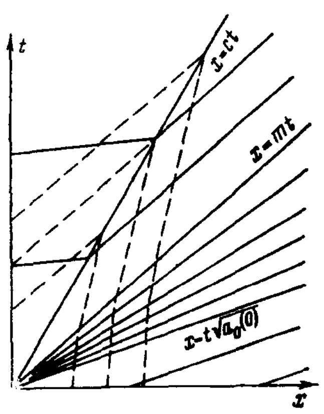

Fig. 3

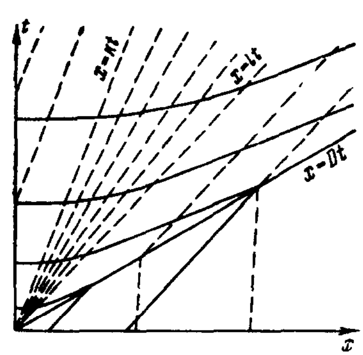

Fig. 4

of the medium we can establish experimentally the functions $\varphi(\varepsilon)$ and $Q(\Gamma)$. To do so it is necessary to vary V_{x0} and V_{y0}.

Variant 2

$$da_0 / d\varepsilon < 0, \ dQ / d\Gamma > 0$$

A centered pure-compression wave propagates through the undisturbed medium; behind this wave there is a zone of constant flow, through which a compression-shear shock-wave travels. Between the shock-wave and the surface there is a zone of constant flow. This case has been studied in [6]. In the xt-plane the motion pattern is as shown in Fig.3. The solution may be written

$$\begin{aligned}
&\varepsilon = 0, \quad \varepsilon_{xy} = 0, \quad V_x = 0, \quad V_y = 0 \quad \text{for } x \geqslant \sqrt{a_0(0)}\, t \\
&x/t = \sqrt{a_0(\varepsilon)}, \quad V_x = -J_1(\varepsilon), \quad \varepsilon_{xy} = 0, \quad V_y = 0 \quad \text{for } \sqrt{a_0(0)}\, t \geqslant x \geqslant mt \\
&\varepsilon = \varepsilon_1, \quad V_x = V_{x1}, \quad \varepsilon_{xy} = 0, \quad V_y = 0 \quad \text{for } mt \geqslant x > Ct \qquad (3.3) \\
&\varepsilon = \varepsilon_2, \quad V_x = V_{x0}, \quad \varepsilon_{xy} = \varepsilon_{xy2}, \quad V_y = V_{y0} \quad \text{for } Ct > x \geqslant 0
\end{aligned}$$

Here

$$J_1(\varepsilon, 0) = \int_0 \sqrt{a_0(\varepsilon)}\, d\varepsilon \qquad (3.4)$$

In order to determine the constants $\varepsilon_1, \varepsilon_2, \varepsilon_{xy2}, V_{x1}, m$ and C we have the system of equations (2.5) to (2.7), to which we must add

$$m = \sqrt{a_0(\varepsilon_1)}, \qquad V_{x1} = -J_1(\varepsilon_1), \qquad X_{x1} = \varphi(\varepsilon_1) + 4Q(|\varepsilon_1|)\varepsilon_1$$

Variant 3

$$da_0 / d|\varepsilon| > 0, \ dQ / d\Gamma < 0$$

A pure-compression shock-wave propagates through the undisturbed medium, followed by a zone of constant flow. Then follows a centered shear wave; between this wave and the surface there is constant flow (Fig. 4). An exact solution cannot be written since the equations of a centered shear-wave (2.10) and (2.1) cannot be integrated. One approximate solution can easily be written if the impulse on the surface of the medium is approximately vertical, i.e. if

$$|V_{y0} / V_{x0}| \ll 1 \qquad (3.4)$$

Making the assumption that the order of φ' is higher than (or equal to) that of Q, and estimating the order of the terms in the equations of the shear-wave, we obtain the following approximate solution to the problem

$$\begin{aligned}
&\varepsilon = 0, \quad \varepsilon_{xy} = 0, \quad V_x = 0, \quad V_y = 0 \quad \text{for } x > Dt \qquad (3.5) \\
&\varepsilon = \varepsilon, \quad \varepsilon_{xy} = 0, \quad V_x = V_{x1}, \quad V_y = 0 \quad \text{for } Dt > x \geqslant lt \\
&x/t = \sqrt{C_0(|\varepsilon|)}, \quad V_x = -J_2(\varepsilon, \varepsilon_1), \quad \varepsilon_{xy} = -\sqrt{2J_3(\varepsilon, \varepsilon_1)} \\
&V_y = J_4(\varepsilon, \varepsilon_1) \quad \text{for } lt \geqslant x \geqslant kt \\
&\varepsilon = \varepsilon_2, \quad \varepsilon_{xy} = \varepsilon_{xy2}, \quad V_x = V_{x0}, \quad V_y = V_{y0} \quad \text{for } kt \geqslant x \geqslant 0
\end{aligned}$$

$$J_2(\varepsilon, \varepsilon_1) = \int_{\varepsilon_1}^{\varepsilon} \sqrt{C_0(|\delta|)}\, d\varepsilon + V_{x1}, \qquad C_0(|\varepsilon|) = \frac{3Q(|\varepsilon|)}{\rho_0} \equiv C \Big|_{\varepsilon_{xy}=0}$$

$$J_3(\varepsilon, \varepsilon_1) = \int_{\varepsilon_1}^{\varepsilon} \frac{d\varphi / d\varepsilon + Q(|\varepsilon|) + 4|\varepsilon| Q'(|\varepsilon|)}{Q'(|E|)} d\varepsilon \qquad (3.6)$$

$$J_4(\varepsilon, \varepsilon_1) = \int_{\varepsilon_1}^{\varepsilon} \frac{\sqrt{C_0(|\varepsilon|)}}{Q'(|\varepsilon|)\sqrt{2J_3(\varepsilon, \varepsilon_1)}} \left(\frac{d\varphi}{d\varepsilon} + Q(|\varepsilon|) + 4|\varepsilon| Q'(|\varepsilon|)\right) d\varepsilon$$

The constants ε_1, ε_2, ε_{xy2}, V_{x1}, D, l and k can be found from Equations (2.4), to which must be added the relations

$$l = C_0\sqrt{(|\varepsilon_1|)}, \qquad k = \sqrt{C_0(|\varepsilon_2|)}, \qquad -J_2(\varepsilon_2, \varepsilon_1) = V_{x0}, \qquad J_4(\varepsilon_2, \varepsilon_1) = V_{y0}$$

$$\varepsilon_{xy2} = -\sqrt{2J_3(\varepsilon_2, \varepsilon_1)} \qquad (3.7)$$

Variant 4

$$da_0 / d|\varepsilon| < 0, \quad dQ / d\Gamma < 0$$

A centered pure-compression wave propagates through the undisturbed medium, followed by a zone of constant flow. Then follows a centered shear-wave between which and the surface there is a zone of constant flow (Fig.5). The solution, when conditions (3.4) are satisfied, is of the form

$$\varepsilon = 0, \quad \varepsilon_{xy} = 0, \quad V_x = 0, \quad V_y = 0 \qquad \text{for } x \geqslant \sqrt{a_0(0)}\, t \qquad (3.8)$$

$$x / t = \sqrt{a_0(\varepsilon)}, \quad V_x = -J_1(\varepsilon), \quad \varepsilon_{xy} = 0, \quad V_y = 0 \qquad \text{for } \sqrt{a_0(0)}\, t \geqslant x \geqslant mt$$

$$\varepsilon = \varepsilon_1, \quad V_x = V_{x1}, \quad \varepsilon_{xy} = 0, \quad V_y = 0 \qquad \text{for } mt \geqslant x \geqslant lt$$

$$x / t = \sqrt{C_0(|\varepsilon|)}, \quad V_x = -J_2(\varepsilon, \varepsilon_1), \quad V_y = J_4(\varepsilon, \varepsilon_1)$$

$$\varepsilon_{xy} = -\sqrt{2J_3(\varepsilon, \varepsilon_1)} \qquad \text{for } lt \geqslant x \geqslant kt$$

$$\varepsilon = \varepsilon_2, \quad \varepsilon_{xy} = \varepsilon_{xy2}, \quad V_x = V_{x0}, \; V_y = V_{y0} \quad \text{for } kt \geqslant x \geqslant 0$$

The constants $\varepsilon_1, \varepsilon_2$, ε_{xy2}, V_{x1}, m, l and k can be found from Equations (3.6), together with the relations

$$m = \sqrt{a_0(\varepsilon_1)}, \qquad V_{x1} = -J_1(\varepsilon_1) \qquad (3.9)$$

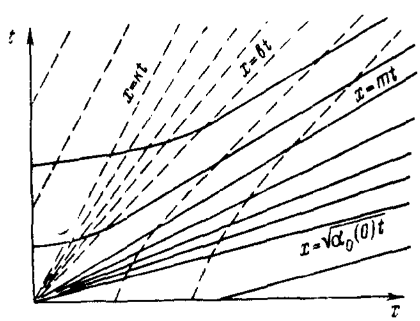

Fig. 5

4. We consider now the experimental determination of the functions $\varphi(\varepsilon)$ and $Q(\Gamma)$. As has already been pointed out, it is possible that a medium exists for which in the case of small deformations $da_0 / d|\varepsilon| > 0$, $dQ / d\Gamma > 0$. For such a medium formulas (3.2) can be used to derive experimental relations $\varphi(\varepsilon)$ and $Q(\Gamma)$ by subjecting a medium (for example, soil) to an oblique impact at the surface. However, although the limitation $da_0/d|\varepsilon| > 0$ will be satisfied (for instance, in experiments of pure compression a shock-wave is fixed in a soil [3 and 4]), the assumption that $dQ/d\Gamma > 0$ is open to dispute. So far no experiments have been carried out on materials, in particular in soils, in shear-compression; therefore the question of whether the shear-compression wave is a shock-wave ($Q' \geqslant 0$) or whether it is continuous ($Q' < 0$) remains unanswered experimentally. Experimental curves drawn from Formulas (3.2) must therefore be verified. This can be done by an experiment in pure compression ($V_y = 0$).

The solution for $da_0/d|\varepsilon| > 0$ (a shock-wave in front followed by constant flow) in the case of pure compression is of the form

$$\varepsilon = 0, \quad V_x = 0, \quad X_x = X_{x0} \quad \text{for } x > Dt \qquad (4.1)$$

$$\varepsilon = \varepsilon_1, \quad V_x = V_{x0}; \quad X_x = X_{x1} \quad \text{for } Dt > x \geqslant 0$$

If V_{x0} is given, then ε_1 and D (the velocity of the shock-wave) can be found from Equations (2.4) in which we set $V_{x1} = V_{x0}$. Remembering that in this case $Y_y = Z_z$ and therefore $\varphi = \frac{1}{3}(X_x + 2Y_y)$, we conclude that experimental curves for $\varphi(\varepsilon)$ and $Q(\Gamma)$ can be drawn if in an experiment of pure compression ($V_y \equiv 0$) we measure V_{x0}, D and Y_{y1} (Y_{y1} is the value of Y_y at the surface of the soil). The solution of the problem is

$$\varepsilon_1 = -\frac{V_{x0}}{D}, \qquad \varphi(\varepsilon_1) = \frac{1}{3}(X_{x0} + 2Y_{y1} - \rho_0 D V_{x0})$$

$$Q(|\varepsilon_1|) = \frac{1}{6}\frac{\rho_0 D V_{x0} - X_{x0} + Y_{y1}}{V_{x0}} D \qquad (4.2)$$

A second check on the accuracy of the results obtained is provided by a comparison of the relations $\varphi(\varepsilon)$, obtained in the manner indicated from Formulas (3.2), with those obtained by the well-known experiment on hydrostatic compression described in [5].

If the function $Q(\Gamma)$ obtained from Formulas (4.2) is found to decay ($Q' < 0$), then the solution (3.1) and Formulas (3.2) are not applicable to the calculation of a shear-compression impact. A check on the validity of experimental curves for $\varphi(\varepsilon)$ and $Q(\Gamma)$ obtained from (4.2) is provided in this case by the solution to the problem of shear-compression impact given by the formulas of variant 3 with experimentally determined parameters (for example, the velocity D of the shock-wave, the stresses X_x, X_y at the surface of the soil). The solution to the problem is then given on the basis of experimental functions $\varphi(\varepsilon)$ and $Q(\Gamma)$.

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